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Nonlinear Elliptic-Parabolic Equations and $\mathcal{B}$-Pseudomonotonicity

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Abstract—Pseudomonotonicity seems to be the good notion to deal with convergence in nonlinear terms of partial differential equations. Lions [1] used two different definitions of pseudomonotonicity for elliptic and parabolic problems, and derived associated existence results. Nonlinear elliptic-parabolic equations are intermediate equations for which an intermediate pseudomonotonicity is defined and an existence result is proved, extending previous results of Alt and Luckhaus [2] and Bermúdez, Durany and Saguez [3].

Keywords—Nonlinear elliptic parabolic equations, Compactness, Pseudomonotonicity, Viscous compressible flow.

1. INTRODUCTION

Let $\mathcal{A}$ and $\mathcal{B}$ be two nonlinear operators over a function space $\mathcal{V}$, with $\mathcal{B}$ possibly multivalued. We consider the following Cauchy problem: given $T > 0$, f and v^0, find u such that

$$\frac{d}{dt}\mathcal{B}(u) + \mathcal{A}(u) \ni f \quad \text{on } [0,T], \qquad \mathcal{B}(u)(0) \ni v^0. \tag{EP}$$

Besides its own mathematical interest, this equation arises quite often in diffusion and free boundary problems. The case where $\mathcal{B}$ is an unbounded linear operator was considered first by Bardos and Brézis [4]. In the nonlinear case, Raviart [5], Grange and Mignot [6], and DiBenedetto and Showalter [7] proved existence results assuming that $\mathcal{A}$ and $\mathcal{B}$ are at least monotone operators, and $\mathcal{B}$ is compact. Alt and Luckhaus [2] investigated the case of a noncompact operator $\mathcal{B}$, assuming $\mathcal{A}$ is strongly monotone. Bermúdez, Durany and Saguez devoted their work to the case where $\mathcal{B}$ is compact and strongly monotone and $\mathcal{A}$ is pseudomonotone. We are interested in the same case excepted that $\mathcal{B}$ is no longer assumed to be strongly monotone; thus the equation may degenerate to an elliptic one.

2. PSEUDOMONOTONICITY

Let V and W be two separable and reflexive Banach spaces, such that V is dense and compactly embedded in W. Denote this injection by $\imath$, and its dual operator by i^*. Let

$$\mathcal{V} = L^p(0,T;V) \quad \text{and} \quad \mathcal{W} = L^p(0,T;W) \qquad \text{where } p \in]1,+\infty[.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let $\mathcal{A}$ be a bounded and coercive operator from $\mathcal{V}$ to $\mathcal{V}'$, and consider the following elliptic problem.

ELLIPTIC PROBLEM. Given $f \in \mathcal{V}'$, find $u \in \mathcal{V}$ such that

$$\mathcal{A}(u) = f. \tag{E}$$

Using a Galerkin method we get an approximate solution u_n weakly convergent to u. The main trick in nonlinear analysis of P.D.E. is to justify the convergence of nonlinear terms, that is

$$\mathcal{A}(u_n) \rightharpoonup \mathcal{A}(u). \tag{1}$$

When (1) cannot be easily proved, one may hope in general to show, using *a priori* estimates, that

$$\limsup(\mathcal{A}(u_n), u_n - u) \leq 0. \tag{2}$$

It is straightforward to see that (1) follows from (2) and from

$$\liminf(\mathcal{A}(u_n), u_n - v) \geq (\mathcal{A}(u), u - v) \qquad \text{for all } v \in \mathcal{V}. \tag{3}$$

As a matter of fact, Lions proved in [1] the existence of a solution to (E) provided $\mathcal{A}$ is pseudomonotone over $\mathcal{V}$; for each sequence u_n weakly converging to u in $\mathcal{V}$, it follows from (2) that (3) holds.

On the other hand, consider the parabolic problem.

PARABOLIC PROBLEM. Given $f \in \mathcal{V}'$ and $u_0 \in V$, find $u \in \mathcal{V}$ such that

$$\frac{du}{dt} + \mathcal{A}(u) = f \qquad \text{and} \qquad u(0) = u^0. \tag{P}$$

In this case, one usually obtains the convergence of u_n', and (P) has a solution according to Lions as soon as $\mathcal{A}$ is assumed to be pseudomonotone over $D\left(\frac{d}{dt}\right)$; for each sequence u_n weakly converging to u in $\mathcal{V}$, such that u_n' weakly converges to u' in $\mathcal{V}'$, it follows from (2) that (3) holds.

Now let $\mathcal{B}$ be a bounded operator from $\mathcal{V}$ to $\mathcal{W}'$. We consider the following elliptic parabolic problem.

ELLIPTIC PARABOLIC PROBLEM. Given $f \in \mathcal{V}'$ and $v_0 \in W'$, find $u \in \mathcal{V}$ solution of (EP).

Trying to find a good definition of pseudomonotonicity for (EP) could lead us to assume that $\mathcal{B}(u_n)'$ weakly converges in $\mathcal{V}'$ instead of having the assumption of the weak convergence of u_n'. However, $\mathcal{B}(u_n)$ does not converge in (or even belong to) $\mathcal{V}$, so we could not get any compactness result of Aubin [1] type. That is why we state this definition.

DEFINITION. *$\mathcal{A}$ is $\mathcal{B}$-pseudomonotone provided that for each sequence u_n weakly convergent to u in $\mathcal{V}$, such that $\mathcal{B}(u_n)$ strongly converges to $\mathcal{B}(u)$ in $\mathcal{W}'$, (2) implies (3).*

3. ASSUMPTIONS AND EXISTENCE RESULT

We assume the following.

(A1) B is a maximal monotone operator from V to V' which is constructed as follows: let Φ be a proper lower semicontinuous convex function on W, finite and continuous on $i(V)$, and such that $\Phi(0) = 0$. We take $B = \partial(\Phi \circ i) = i^* \circ \partial\Phi \circ i$.

(A2) $\exists C > 0$, $\forall u \in V$, $\forall v \in \partial\Phi \circ i(u)$, $\|v\|_{W'} \leq C(1 + \|u\|_V^{p-1})$. Defining $\mathcal{B} : \mathcal{V} \to \mathcal{V}'$ by $v \in \mathcal{B}(u) \Leftrightarrow v(t) \in B(u(t))$ a.e. on $[0,T]$, $\mathcal{B}$ is a bounded maximal monotone operator from $\mathcal{V}$ to $\mathcal{W}'$.

Next we consider a family of operators $\{A(t,.),\ t \in [0,T]\}$ from V to V' satisfying

(A3) $\exists C > 0$, $\forall u \in V$, $\|A(t,u)\|_{V'} \leq C(1 + \|u\|_V^{p-1})$ a.e. on $[0,T]$.

(A4) Defining $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ by $\forall u \in \mathcal{V}$, $\mathcal{A}(u)(t) = A(t, u(t))$ a.e. on $[0,T]$, we have

$$\liminf_{\|u\|_{\mathcal{V}} \to \infty} \frac{(\mathcal{A}(u), u)}{\|u\|_{\mathcal{V}}^p} > 0.$$

Now we state our main result.

THEOREM 1. *Let $f \in \mathcal{V}'$, $u^0 \in D(B)$ and $v^0 \in B(u^0)$ be given. Under assumptions (A1)-(A4), if either $\mathcal{A}$ is pseudomonotone on $\mathcal{V}$ or only $\mathcal{B}$-pseudomonotone and $\mathcal{B}$ is continuous over $\mathcal{W}$ for the strong topology, there exist $u \in \mathcal{V}$ and $v \in \mathcal{W}' \cap L^\infty(0,T;V')$ such that $\frac{dv}{dt} \in \mathcal{V}'$ and*

$$\frac{dv}{dt} + \mathcal{A}(u) = f, \qquad v \in \mathcal{B}(u), \quad v(0) = v^0.$$

Moreover, if Φ is continuous over W, we also get $v \in L^\infty(0,T;W')$.

4. OUTLINE OF THE PROOF

The proof of Theorem 1 uses the same time discretisation process as in [3], leading to an approximate problem which is an elliptic variational inequality solved in [1]. Note that the proof of Lemma 1 in [3] is even easier for a $\mathcal{B}$-pseudomonotone operator, because we do not have to ensure the convergence of the time derivatives.

Denoting by u_ε and $v_\varepsilon \in \mathcal{B}(u_\varepsilon)$ the step functions associated to the time discretisation, and $\hat{v}_\varepsilon$ the corresponding affine function, we obtain the following *a priori* estimates: $u_\varepsilon \in \mathcal{V}$, $v_\varepsilon \in \mathcal{W}'$, $\mathcal{A}(u_\varepsilon) \in \mathcal{V}'$, and $\frac{d\hat{v}_\varepsilon}{dt} \in \mathcal{V}'$ are bounded independently of ε. We also obtain that $(\Phi \circ i)^*(v_\varepsilon)$ is bounded in $L^\infty(0,T)$ and thus v_ε is bounded in $L^\infty(0,T;V')$.

Extracting subsequences, we get the corresponding weak convergences. Using a compactness lemma [5, Lemma 1.4] we obtain the strong convergence of v_ε in $\mathcal{V}'$. This result combined with the maximal monotonicity of $\mathcal{B}$ allows us to pass to the limit in the nonlinear term $\mathcal{B}(u_\varepsilon)$. If $\mathcal{A}$ is assumed to be pseudomonotone over $\mathcal{V}$ then we complete the proof using a chain rule as in [3].

If $\mathcal{A}$ is only assumed to be $\mathcal{B}$-pseudomonotone, with $\mathcal{B}$ continuous from $\mathcal{W}$ to $\mathcal{W}'$ for the strong topologies (this means also that B is continuous from W to W' and compact from V to W'), then we have to prove the strong convergence of v_ε in $\mathcal{W}'$. To this aim, following Simon [8] we obtain uniform estimates on translations $\tau_h v_\varepsilon(t) = v_\varepsilon(t+h)$ for $t \le T-h$ and 0 for $t > T-h$, and prove the following.

THEOREM 2. *Let V and W be reflexive and separable Banach spaces such that V is compactly embedded in W. Let E be a compact operator from V to W' such that its extension $\mathcal{E}$ is bounded from $L^p(0,T;V)$ to $L^{p'}(0,T;W')$. Consider a bounded family $\{u_\varepsilon\} \subset L^p(0,T;V)$ and let $\{v_\varepsilon\}$ such that $v_\varepsilon = \mathcal{E}(u_\varepsilon)$. Then if*

$$\lim_{h\to 0} \|\tau_h v_\varepsilon - v_\varepsilon\|_{L^{p'}(0,T-h;W')} = 0 \qquad \text{uniformly in } \varepsilon, \tag{4}$$

$\{v_\varepsilon\}$ is relatively compact in $L^{p'}(0,T;W')$.

PROOF. We approximate v_ε by step functions and following Alt and Luckhaus [2], we prove the compactness of the family

$$\left\{\int_{t_1}^{t_2} v_\varepsilon(t)\,dt\right\}_{\{\varepsilon>0\}} \qquad \text{in } W'.$$

Then we use a compactness result from Simon [8, Theorem 1]. ∎

It remains to prove (4) which is obtained thanks to the following lemma.

LEMMA 1. *Assume B is continuous from W to W'; then (4) holds provided that:*

(i) $u_\varepsilon \rightharpoonup u$ *weakly in* $\mathcal{V}$,

(ii) $\exists C > 0$, $\forall \varepsilon > 0$, $\forall h \in]0,T[$, $\int_0^{T-h} (v_\varepsilon(t+h) - v_\varepsilon(t), u_\varepsilon(t+h) - u_\varepsilon(t))_H \, dt \leq Ch^{1/p}$,

(iii) $\exists C > 0$, $\forall \varepsilon > 0$, $\operatorname{ess\,sup}_{t\in[0,T]} \Phi^* (v_\varepsilon(t)) \leq C$.

Note that (iii) is used to get a bound for v_ε in $L^\infty(0,T;W')$.

5. APPLICATION

We are interested in the modelisation of the injection moulding of a viscous compressible fluid, under the *Hele Shaw* assumption. In this assumption [9, p. 38], the pressure is uniform in the mold thickness, and the velocity can be computed directly from the pressure and the fluid viscosity. Thus our attention focussed on the pressure equation, which may be stated as follows. Let N, p, q be three integers with $N > 0$, $p > q > 1$, Ω an open subset of $\mathbb{R}^N$ which stands for the mold. The fluid flows into the mold through $\Gamma_1 \subset \partial\Omega$, for $T > 0$ units of time, with a time dependent flux denoted by $g \in L^{p'}(\Sigma_1)$. Here $\Sigma_1 = \Gamma_1 \times]0,T[$, and $Q = \Omega \times]0,T[$. At the same time the air which initially filled the mold gets out through Σ_2 at constant pressure, assumed to be zero. A forcing term $f \in L^{p'}(Q)$ accounts for other physical aspects. Our working space V is a closed subspace of $W^{1,p}(\Omega)$ containing $W_0^{1,p}(\Omega)$ and defined by

$$V = \left\{ u \in W^{1,p}(\Omega),\ u_{|\Gamma_2} = 0 \right\}.$$

The pressure equation in u is

$$\begin{aligned}
\frac{\partial}{\partial t} \beta\,(x,u) - \operatorname{div} \vec{A}\,(x,t,u,\beta(u),\nabla u) + A_0\,(x,t,u,\beta(u),\nabla u) &= f, \\
\vec{A}\,(x,t,u,\beta(u),\nabla u)\,.\vec{n} &= g \qquad \text{on } \Sigma_1, \\
u &= 0 \qquad \text{on } \Sigma_2, \\
\beta\,(x,u(x,0)) &= \beta\left(x,u^0(x)\right) \qquad \text{a.e. on } \Omega.
\end{aligned}$$

In that equation $\beta(x,.)$ is assumed to be a continuous and strictly increasing function over $\mathbb{R}$ subject to the following growth condition:

$$\exists c_1,\, c_2 > 0, \quad \forall r \in \mathbb{R}, \qquad |\beta\,(x,r)\,| \leq c_1 r^{p-1} + c_2, \qquad \text{a.e. in } \Omega.$$

For the elliptic part, $A_i(x,t,\eta,\nu,\xi)$, $i \in \{0,\dots,N\}$ is a family of real valued functions, defined over $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and verifying:

(A1) For almost every $(x,t) \in Q$, $(\eta,\nu,\xi) \mapsto A_i(x,t,\eta,\nu,\xi)$ is continuous over $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and for all (η,ν,ξ), $(x,t) \mapsto A_i(x,t,\eta,\nu,\xi)$ is measurable on Q.

(A2) For all (u,v,w) belonging to $L^q(Q) \times L^{p'}(Q) \times (L^p(Q))^N$, $A_i(x,t,u,v,w)$ belongs to $L^{p'}(Q)$ for $i \in \{1,\dots,N\}$ and to $L^{q'}(Q)$ for $i = 0$.

(A3) The vector function $\vec{A} = (A_i)_{1 \leq i \leq N}$ verifies

$$\lim_{|\xi| \to +\infty} \frac{\left(\vec{A}(x,t,\eta,\nu,\xi),\xi\right)}{|\xi| + |\xi|^{p-1}} = +\infty,$$

for almost all fixed x, t in Q and bounded $|\eta|$, $|\nu|$, and

$$\left(\vec{A}(x,t,\eta,\nu,\xi) - \vec{A}(x,t,\eta,\nu,\xi^*)\right).(\xi - \xi^*) > 0 \qquad \text{if } \xi \neq \xi^*$$

almost everywhere on Q and for all η, ν.

These equations are not in the framework of known existence theorems about doubly nonlinear equations because the nonlinearity $\beta(x,.)$ is *not strongly monotone* and u appears *explicitly* in the elliptic part and in A_0.

Theorem 2 enables us to get strong and thus a.e. convergence of $\beta(x, u_\varepsilon)$, which thanks to the strict monotonicity of β gives us a.e. convergence of u_ε, and thus strong convergence in $L^q(Q)$ for $q < p$. This convergence enables us to pass to the limit in nonlinear terms of the equation, as soon as suitable growth assumptions are made.

The detailed proof of the $\mathcal{B}$-pseudomonotonicity of $\mathcal{A}$ is tedious and will be given in [10].

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